Double Factorization Systems and Double Fibrations

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Orthogonal Factorization Systems (OFS)

An OFS for a category $\mathcal{C}$ is a pair of classes $(\mathcal{L}, \mathcal{R})$ of morphisms in $\mathrm{Mor}(\mathcal{C})$ such that:

1. for every morphism f, there are two morphisms $l \in \mathcal{L}$ and $r \in \mathcal{R}$ such that $f = r \circ l$

$$\begin{tikzcd} \bullet \arrow[rr, "f"] \arrow[dr, "l\in\mathcal{L}"'] & & \bullet \\ & \bullet \arrow[ur, "r\in\mathcal{R}"'] & \end{tikzcd}$$

2. The classes are closed under having the lifting property against each other:

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "l\in\mathcal{L}"'] & \bullet \arrow[d, "r\in\mathcal{R}"] \\ \bullet \arrow[r, "g"'] \arrow[ur, dashed, "\exists!"] & \bullet \end{tikzcd}$$

Functorial Factorizations

A *functorial factorization* for $\mathcal{C}$ is an assignment, to each morphism $f \in \mathrm{Mor}(\mathcal{C})$, of a pair of morphisms l_f, r_f such that $f = r_f \circ l_f$, and this assignment extends to a functor $Q \colon \mathcal{C}^2 \to \mathcal{C}$ (where $\mathcal{C}^2$ is the arrow category of $\mathcal{C}$) sending every f to the object in the middle of the factorization of f.

$$
\begin{tikzcd}
\bullet \arrow[r, "l_f"] \arrow[rr, bend left, "f"] \arrow[d, "u"'] & Q(f) \arrow[r, "r_f"] \arrow[d, dashed, "\exists!"', "Q(u,v)"] & \bullet \arrow[d, "v"] \\
\bullet \arrow[r, "l_g"'] \arrow[rr, bend right, "g"'] & Q(g) \arrow[r, "r_g"'] & \bullet
\end{tikzcd}
$$

Note that every OFS is a functorial factorization system.

Remarks on OFS

1. Normal pseudo-algebras for the 2-monad $(-)^2 : \mathrm{Cat} \to \mathrm{Cat}$ are categories with an OFS [Korostenski &Tholen JPAA 1993].
2. We can look at morphisms between algebras for this 2-monad to get morphisms between categories with an OFS.

Double categories

Double categories are categories with two types of morphisms (arrows, drawn horizontally, and proarrows, drawn vertically) and double cells such that vertical and horizontal compositions are compatible.

Examples:

1. $\mathbb{S}\mathrm{pan}(\mathcal{C})$: arrows are morphisms of $\mathcal{C}$ and proarrows are spans.
2. $\mathbb{R}\mathrm{el}(\mathrm{Set})$: arrows are functions and proarrows are relations
3. $\mathbb{Q}(\mathcal{C})$: arrows and proarrows are morphisms of $\mathcal{C}$
4. $\mathbb{M}\mathrm{od}$: objects are rings with units, arrows are ring morphisms, proarrows are bimodules, and cells are equivariant morphisms of groups.

They can also be seen as internal categories in Cat: $\mathbb{X}_0$ is the category of objects and arrows, $\mathbb{X}_1$ is the category of proarrows and double cells.

DOFS on Double Categories

- The goal of this part of the talk is to explore the ways in which one can define a **double orthogonal factorization system** on a double category.
- This will involve factorizations of both arrows and double cells with respect to horizontal composition.
- Let's first consider two motivating examples.

Examples: a DOFS on $\mathbb{S}\mathrm{pan}(\mathcal{C})$

Let $(\mathcal{L}, \mathcal{R})$ be an OFS for $\mathcal{C}$. Then $\mathbb{S}\mathrm{pan}(\mathcal{C})$ has a DOFS as follows:

$$
\begin{tikzcd}
\bullet \arrow[r, "l_1"] & \bullet \arrow[r, "r_1"] & \bullet \\
\bullet \arrow[u] \arrow[d] \arrow[r, Rightarrow, "l_3"] & \bullet \arrow[u, dashed] \arrow[d, dashed] \arrow[r, Rightarrow, "r_3"] & \bullet \arrow[u] \arrow[d] \\
\bullet \arrow[r, "l_2"'] & \bullet \arrow[r, "r_2"'] & \bullet
\end{tikzcd}
$$

1. Left class of double cells are arrows from $\mathcal{L}$ between proarrows.
2. Right class of double cells are arrows from $\mathcal{R}$ between proarrows.

Source, target, and identities all preserve both classes of the factorization. Vertical multiplication only preserves the right class in general, and preserves factorizations laxly.

Examples: a DOFS on $\mathbb{R}\mathrm{el}(\mathrm{Set})$

The double category $\mathbb{R}\mathrm{el}(\mathrm{Set})$ inherits factorizations of both arrows and double cells from the epi-mono factorization system on Set:

$$
\begin{tikzcd}
S \arrow[rr, two heads] \arrow[rrrr, bend left, "f"] \arrow[dd, "R"', "|" marking] & & f(S) \arrow[rr, hook] \arrow[dd, "(f \times g)(R)"', "|" marking] & & S' \arrow[dd, "R'", "|" marking] \\
& \arrow[r, Rightarrow, two heads] & & \arrow[r, Rightarrow, hook] & \\
T \arrow[rr, two heads] \arrow[rrrr, bend right, "g"'] & & g(T) \arrow[rr, hook] & & T'
\end{tikzcd}
$$

The unit and the source and target maps preserve the two classes of arrows strictly.

DOFS on $\mathbb{R}\mathrm{el}(\mathrm{Set})$ - continued

However, vertical composition does not preserve the left class and preserves factorizations laxly.

$$
\begin{tikzcd}
\{s_0\} \arrow[r, two heads] \arrow[d, "R=\{(s_0,t_0)\}"] \arrow[dd, bend right, "\emptyset"'] & \{*\} \arrow[d, "\{(*,*)\}"'] \arrow[dd, bend left, "\{(*,*)\}"] \\
\{t_0, t_1\} \arrow[r, two heads] \arrow[d, "R'=\{(t_1,s_1)\}"] & \{*\} \arrow[d, "\{(*,*)\}"'] \\
\{s_1\} \arrow[r, two heads] & \{*\}
\end{tikzcd}
$$

The vertical composition is a lax morphism between categories with an OFS, viewed as algebras for the monad $(-)^2$.

Strict and lax DOFS as internal categories

A **(strict) double orthogonal factorization system** DOFS for a double category $\mathbb{D}$ is an internal category in categories with OFS and functors preserving the left and the right class, this means:

- an OFS on the category $\mathbb{D}_0$ (factorization of arrows)
- an OFS on the category $\mathbb{D}_1$ (factorization of double cells)

compatible with each other.
We ask vertical source, target, identity and composition to strictly preserve factorizations.

A **lax double orthogonal factorization system** is an internal category in categories with OFS and lax morphisms between them. We still ask vertical source, target and identity to strictly preserve the factorizations. Vertical composition will now be a lax morphism of algebras.

Double Fibrations (Another Source of Examples)

A **double fibration** is a (strict) double functor $P : \mathbb{E} \to \mathbb{B}$ between (pseudo) double categories

$$\begin{tikzcd}
E_1 \times_{E_0} E_1 \arrow[r, "\otimes_E"] \arrow[d, "P_1 \times_{P_0} P_1"'] & E_1 \arrow[r, shift left=2, "\mathrm{src}_E"] \arrow[r, shift right=2, "\mathrm{tgt}_E"'] \arrow[d, "P_1"'] & E_0 \arrow[l, "y_E" description] \arrow[d, "P_0"] \\
B_1 \times_{B_0} B_1 \arrow[r, "\otimes_B"'] & B_1 \arrow[r, shift left=2, "\mathrm{src}_B"] \arrow[r, shift right=2, "\mathrm{tgt}_B"'] & B_0 \arrow[l, "y_B" description]
\end{tikzcd}$$

such that

1. P_0 and P_1 are fibrations,
2. they admit a cleavage such that src_E and tgt_E are cleavage-preserving, and
3. y_E and $\otimes_E$ are Cartesian-morphism preserving.

Lifting DOFS along double fibrations

Suppose $p : \mathcal{E} \to \mathcal{B}$ is a fibration of 1-categories, and $\mathcal{B}$ has an OFS. Then, the OFS on $\mathcal{B}$ can be lifted to one on $\mathcal{E}$.

We have an analogous result in the double setting:

Theorem (Double Lifting)

Given a double fibration $p : \mathbb{E} \to \mathbb{B}$, a (strict) DOFS on $\mathbb{B}$ can be lifted to one on $\mathbb{E}$.

- Lifting the trivial DOFS on $\mathbb{B}$, we obtain the (pseudo-vertical, cartesian) DOFS on $\mathbb{E}$.

We conjecture a similar result to the double lifting theorem for lax DOFS.

The Arrow Monad on DblCat

Consider the 2-monad $T = (-)^{\rightarrow} : \mathrm{DblCat} \to \mathrm{DblCat}$ which sends a double category $\mathbb{A}$ to the double category $\mathbb{A}^{\rightarrow}$ with

- **Objects**: arrows in $\mathbb{A}$;
- **Arrows:** commutative squares in the arrows of $\mathbb{A}$;
- **Proarrows:** double cells in $\mathbb{A}$;
- **Double Cells**: commutative squares of double cells (in the arrow direction).

Two Versions

The arrow monad is a 2-monad on both

- the 2-category $\mathrm{DblCat}_{\mathsf{ps}}$ of double categories, normal pseudo double functors and horizontal transformations
- the 2-category $\mathrm{DblCat}_{\mathsf{lax}}$ of double categories, normal lax double functors and horizontal transformations

Theorem

- *Normal pseudo algebras for the arrow monad on* $\mathrm{DblCat}_{\mathsf{ps}}$ *are strict DOFS;*
- *Normal pseudo algebras for the arrow monad on* $\mathrm{DblCat}_{\mathsf{lax}}$ *are lax DOFS.*

The graphical structure of a functor, part 1

For a functor $p : \mathcal{E} \to \mathcal{B}$, given an object $b \in \mathcal{B}$ and an object $a \in \mathcal{E}$ such that $p(a) = b$, we think of b as lying *over* a, and depict this as follows:

$$\begin{tikzcd} a \arrow[d, mapsto, dotted] \\ b \end{tikzcd}$$

Similarly, given a morphism $g : d \to b \in \mathcal{B}$ and $f : c \to a \in \mathcal{E}$ such that $p(a) = b$, $p(c) = d$, and $p(f) = g$, we think of f as lying over g, and depict this as a *square*:

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, mapsto, dotted] & a \arrow[d, mapsto, dotted] \\ d \arrow[r, "g"'] & b \end{tikzcd}$$

The graphical structure of a functor, part 2

Squares of this form can be composed horizontally (by the fact that p must preserve composition of morphisms).

$$
\begin{tikzcd}
s \arrow[r, "h"] \arrow[d, dotted, maps to] & c \arrow[r, "f"] \arrow[d, dotted, maps to] & a \arrow[d, dotted, maps to] \\
t \arrow[r, "k"'] & d \arrow[r, "g"'] & b
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
s \arrow[r, "f \circ h"] \arrow[d, dotted, maps to] & a \arrow[d, dotted, maps to] \\
t \arrow[r, "g \circ k"'] & b
\end{tikzcd}
$$

When the composition of two such squares is equal to a third as above, these naturally arrange themselves into a *triangular prism*:

$$
\begin{tikzcd}
s \arrow[dr, "h"'] \arrow[drr, "f \circ h"] \arrow[dd, dotted, maps to] & & \\
& b \arrow[r, "f"'] \arrow[dd, dotted, maps to] & a \arrow[dd, dotted, maps to] \\
t \arrow[dr, "k"'] \arrow[drr, "g \circ k"] & & \\
& d \arrow[r, "g"'] & b
\end{tikzcd}
$$

Cartesian morphisms & fibrations

An immediate upshot of this representation of functors is its ability to straightforwardly capture the structure of *Cartesian morphisms*:

$$
\begin{tikzcd}
c \arrow[r, "f"] \arrow[d, dotted] \arrow[dr, phantom, "\lrcorner" very near start] & a \arrow[d, dotted] \\
d \arrow[r, "g"'] & b
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{tikzcd}
s \arrow[dd, dotted] \arrow[dr, dashed, "\exists!"'] \arrow[drr, "\forall i"] & & \\
& c \arrow[r, "f"'] \arrow[dd, dotted] & a \arrow[dd, dotted] \\
t \arrow[dr, "\forall k"'] \arrow[drr, "\forall j"] & & \\
& d \arrow[r, "g"'] & b
\end{tikzcd}
$$

This graphical framework is thus well-suited to the study of *fibrations*, in at least the 1-categorical case, but in fact for double categories and other higher-categorical structures as well.

From prisms to simplices

Being built upon abstract shapes such as *triangles, prisms, etc.*, our graphical framework is naturally interpreted in the internal language of the topos of *simplicial sets*.

We make use of Riehl & Shulman's development of *simplicial type theory* in order to formalize this framework and results following from it. We have begun formalizing our results in Agda, based on these ideas.

In particular, simplicial type theory gives us two distinct ways to *internally* define categories, either:

- as in ordinary type theory/set theory, or
- using the ambient *simplicial* structure.

Combining these two definitions gives a natural internal definition of *double categories* in simplicial type theory. We hope to make use of this in our ongoing study of double fibrations, double factorization systems, etc.

Work in Progress

- How are DOFS on the double category $\mathbb{Q}(\mathcal{C})$ of quintets related to OFS on 2-categories (Stefan Milius' thesis)?
- Do the OFSs in the category of unitary rings and the category of abelian groups extend to $\mathbb{M}\mathrm{od}$?
- Work out the details on the arrows between double categories with DOFS.
- Study the interaction between the monad for double fibrations and the monads for DOFS.
- Characterize DOFS that are the vertical-Cartesian DOFS of a double fibration.
- So far we have formalized internal categories, double categories, and fibrations in simplicial Agda. We would like to also formalize double fibrations and double factorization systems, and formally verify some of our above results, along with others, such as a characterization of the double Grothendieck construction as a lax colimit.